Sulle medie multiple di potenze

What follows is a fairly literal translation of Bonferroni's paper on multiple algebraic means[1] It was given at the third conference of the Italian Mathematical Union in September 1948 at Pisa.

I have left his italics, and added a single footnote to mark a misprint in the original.

Abstract Defining the multiple means $M_{p,q}$ of the n quantities x_i as the $p+q+\ldots$th root of the mean of cross-products of powers of degree $p, q, \ldots,$ there can be extended to them some properties of the usual algebraic means; it is demonstrated that as two indexes p, q approach one another such as to leave $p+q$ constant the $M_{p,q}$ decreases (for x_i not constant), and it is noted that the ordering by index of the means of the same degree does not always correspond to ordering in magnitude (as is sometimes claimed) because it is necessary to take into account also the variation of multiplicity.

1 The usual algebraic mean of order or degree p of $x_1 \ldots x_n$ (positive), defined as

$$M_p = \sqrt[p]{\frac{S_p}{n}}, \quad S_p = x_1^p + \ldots + x_n^p$$

can be generalised to double means, or of multiplicity 2, of partial degree p, q and total $p+q$, defined as

$$M_{p,q} = \sqrt[p+q]{\frac{S_{p,q}}{n(n-1)}}, \quad S_{p,q} = x_1^p x_2^q + x_2^p x_1^q + \ldots$$

to triple or of multiplicity 3

$$M_{p,q,r} = \sqrt[p+q+r]{\frac{S_{p,q,r}}{n(n-1)(n-2)}}, \quad S_{p,q,r} = x_1^p x_2^q x_3^r + x_2^p x_1^q x_3^r + \ldots$$

and so on for higher multiplicity

2 It is known that M_p is an increasing function of p. This property can be extended in two ways:

a) $M_{p,p,p,\ldots}$ is an increasing function of p; b) $M_{p,q,r,\ldots}$ is an increasing function of its maximum index. This can be shown by differentiating with respect to p and taking into account the properties of concave functions.

3 Similarly it can be shown that under quite wide conditions $M_{p,q,r,\ldots}$ tends to $\sqrt[n]{x_1 \ldots x_n}$ when p, q, r, … tend to zero (a well known property of M_p).

4 The following is an interesting property: if $p > q$ and $p - h > q$ then $(h > 0)$

$$M_{p,q,r,s\ldots} \leq M_{p-h,q+h,r,s,\ldots}$$

The equality holds only for $x_1 = x_2 = \ldots$. Since it is sufficient to show that the property holds for $M_{p,q}$ and for two quantities x_1, x_2, consider

$$S_{p,q} = x_1^p x_2^q + x_2^p x_1^q$$

Letting p and q vary such that $p + q$ is held constant, we have $dq = -dp$ and

$$dS_{p,q} = x_1^q x_2^q (x_1^{p-q} - x_2^{p-q})(\log x_1 - \log x_2) dp$$

so $dS_{p,q}$ has the same sign as dp (if $x_1 \neq x_2$) as long as $p > q$ and the opposite sign if $p < q$. This means that $S_{p,q}$ decreases as p decreases to $\frac{1}{2}(p+q)$ and then increases to the original value (when p is exchanged with q)

Instead of differentiating it can be seen that

$$x_1^p x_2^q + x_2^p x_1^q - (x_1^{p-h} x_2^{q+h} + x_2^{p-h} x_1^{q+h}) = x_1^q x_2^q (x_2^h - x_1^h)(x_2^{p-h-q} - x_1^{p-h-q})$$

and that this difference is positive if $p - h > q$, leading to the same conclusions.

In particular since

$$M_p = M_{p,0} = M_{p,0,0} \quad \text{etc}$$

$$M_{p,q} = M_{p,q,0} = M_{p,q,0,0} \quad \text{etc}$$

then[1]

$$M_p \geq M_{p-h,h}; \quad M_{p,q} \geq M_{p-h,q,h}; \quad \text{etc}$$

5 The preceeding theorems allow us to establish (for integral p, q) the ordering of the means for the first few degrees (for the same x_i). Let us form a table arranging in the same column the means of the same (total) degree: for degree 3 for example these are M_3, $M_{2,1}$, $M_{1,1,1}$. Let us put them in order

[1] Translator's note: the inequality is correct here and reversed in the first formula of this section

1	2	3	4	5	6	7
	11	21	31	41	51	61
			22	32	42	52
		111	211	311	411	511
					33	43
				221	321	421
			1111	2111	3111	4111
						331
					222	322
					2211	3211
				11111	21111	31111
						2221
						22111
					111111	211111
						1111111

of decreasing first index, for equality of first index in order of decreasing second index and so on. The number of means of degree k is the number of partitions of k into k integers (positive or zero) not increasing. In the same row we arrange means in order of increasing first index. Restricting consideration to the first 7 degrees we have the schematic table.

The means in the same row form an increasing series from left to right (by the property from section 2); those in the same column decrease from top to bottom for the first five degrees as each mean is transformed into the next by bringing two indexes closer together (by the property in section 4). Thus

$$M_5 = M_{5,0} > M_{4,1} > M_{3,2} = M_{3,2,0} > M_{3,1,1} > M_{2,2,1} = M_{2,2,1,0} > M_{2,1,1,1} > M_{1,1,1,1,1}$$

For degree 6 however it is not possible to move from $M_{4,1,1}$ to $M_{3,3}$ by bringing closer indexes and we cannot apply the property from 4. Effectively *it is not always the case that* $M_{4,1,1} > M_{3,3}$ (for distinct x_i). It is enough to note that if these two quantities are formed from three values x_1, x_2, x_3, then letting x_1 alone tend to zero will make $M_{4,1,1} < M_{3,3}$ (as when $x_1 = 0$ only the first mean is reduced to zero); while if x_1 alone is allowed to increase sufficiently then $M_{4,1,1} > M_{3,3}$ (because x_1 is involved as degree 4 in $M_{4,1,1}$ and degree 3 in $M_{3,3}$).

The same is true for $M_{3,1,1,1}$ and $M_{2,2,2}$ as can be seen by calculating them for four values and also for the means of degree 7 underlined in the table (compared with the following mean); and in general for the means whose multiplicity is greater than that of succeeding means. Hence: *the ordering* *by indexes constitutes ordering of size only for the first 5 degrees*[2]

From degree 6 onwards the difference between two successive means can change sign. This certainly happens (as has been seen) when the multiplicity decreases; but it can also happen in other cases (for example $M_{5,2,2}$ and $M_{4,1,1}$. For the difference not to change sign it is sufficient that on moving from one mean to another by bringing closer indexes the multiplicity does not diminish.

References

[1] C E Bonferroni. Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana*, 5 third series:267–270, 1950.

[2] G. Zappa, studying certain classes of mean (Osservazioni sopra le medie combinatorie, Metron, Vol XIV, n1, 15-VI-1940, p31) has separated them into several types corresponding precisely to the multiple means studied here. After having verified that $S_{3,2} \geq S_{3,1,1}$ he claimed the demonstration 'of a general character, valid for any two consecutive types and for any order' (p39). The consideration developed here shows, however, that it is not sufficient to take account of ordering by indexes, but that account must be taken as well of the variation in multiplicity.